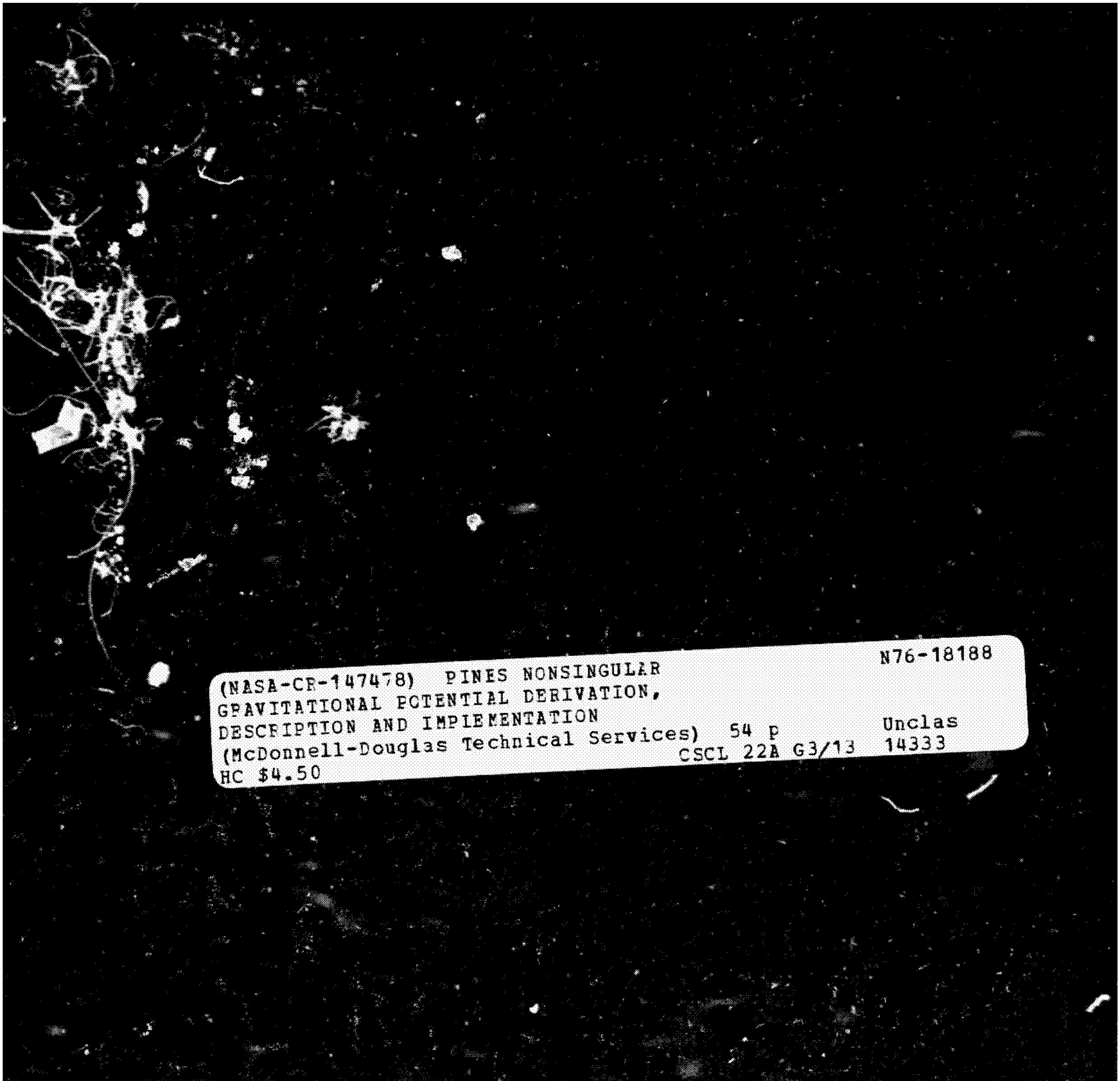

(NASA-CR-147478) PINES NONSINGULAR GRAVITATIONAL POTENTIAL DERIVATION, DESCRIPTION AND IMPLEMENTATION (McDonnell-Douglas Technical Services) 54 p HC $4.50 CSCL 22A G3/13

N76-18188

Unclas 14333

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PINES' NONSINGULAR GRAVITATIONAL POTENTIAL DERIVATION, DESCRIPTION AND IMPLEMENTATION

9 FEBRUARY 1976 MDC W0013

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CONTRACT NAS 9-13970

This report was prepared by the McDonnell Douglas Technical Services Company, Inc., under contract NAS 9-13970, Space Shuttle Engineering and Operations Support, for the Johnson Space Center of the National Aeronautics and Space Administration, Reginald M. Machell, Technical Manager. This work was administered under the technical direction of the Mathematical Physics Branch, Mission Planning and Analysis Division, Data Systems and Analysis Directorate with James C. Kirkpatrick as Technical Monitor.

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PINES' NONSINGULAR GRAVITATIONAL POTENTIAL DERIVATION, DESCRIPTION AND IMPLEMENTATION

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1.0 SUMMARY

The possibility that the shuttle orbiter may be required to go into polar orbits implies the need to derive a representation of the gravitational potential that avoids the usual singularity at the pole. Such a representation, including spherical harmonics coefficients up to any order and degree, has been proposed by S. Pines in "Uniform Representation of the Gravitational Potential and its Derivatives."

The present note contains an engineering interpretation of and some minor corrections to the aforementioned report by Pines. The physical meaning of the variables used by Pines is explained, the derivation of results is separated into smaller parts for easier reading, some additional recurrence relations for the "derived" Legendre polynomials are included and compared, and a computer program implementing this formulation is presented.

Numerical experiments conducted show that the use of this representation, besides satisfying the requirement (removing the singularity), substantially increases the speed of the computation.

2.0 INTRODUCTION

The space shuttle is being designed with a view to its performing a multiplicity of tasks. Some of these may require that it be placed in a polar orbit.

The existence of singularities in polar orbits due to the usual formulation of the gravitational potential makes it necessary that both ground and onboard software developed to support shuttle orbits use a representation of the potential that is free from such singularities. Pines, in reference 1, presents an alternative formulation that satisfies this requirement.

The recursive algorithms proposed by Pines are stable at any order, easy to program, and numerically efficient. They are therefore excellent for both ground software (where a high order may be used) and onboard software (where an adequately truncated version may be obtained simply by stipulating a lower order).

The order that must be used to meet the requirements of the onboard program in terms of size, speed, and accuracy is subject to study, it is possible that different orders may be needed for short-term and long-term state propagation. The ease with which the order may be changed and the variational equations obtained with Pines' representation of the potential make it ideal for such a study.

The formulation by Pines may, then, prove to be very important for both the ground and the flight navigation software. For this reason, the present note has been written to provide (a) an engineering interpretation of reference 1, (b) a preliminary computer program utilizing the algorithms developed by Pines, and (c) a brief report on numerical results.

3.0 DISCUSSION

In this section, the conventional representation of the potential is presented and the nature of the singularity in polar orbits is explained.

The change of variables proposed in reference 1 is then described, and all the phases of the derivation are expanded; the advantages of the new representation are pointed out.

It is shown how the various terms that comprise the new representation can be obtained recursively. For the case of the "derived" Legendre polynomials, where there is more than one way to obtain recurrence relations, various methods are exhibited and compared. One of these methods is recommended: Any errors that may be present in those terms from which a new term is derived are attenuated in the process.

Final expressions for the potential and the gravitational force are collected in the last subsection for easier reference.

3.1 Statement of the Problem

The gravitational potential of a celestial body is commonly given in spherical harmonics. The expression is

$$\phi = \frac{\mu}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n [J_n P_n(\sin\alpha) - \sum_{m=1}^{n} P_{n,m}(\sin\alpha)(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda)]\right\} \tag{1}$$

where

μ is the gravitational constant of the celestial body;

a is the radius (usually the mean radius) of the celestial body;

r is the distance between the origin of the coordinate system and the point B, where the potential is being evaluated;

α is the latitude of point B;

λ is the longitude of point B;

J_n are the zonal harmonics coefficients;

$C_{n,m}$ and $S_{n,m}$ are the coefficients of the tesseral (including the sectorial) harmonics;

P_n are the Legendre polynomials; and

$P_{n,m}$ are the associated Legendre functions.

A derivation of equation (1) may be found, for instance, in reference 2.

The choice of origin and of coordinate axes is very important in this representation. If the origin is chosen at the celestial body's center of mass, the constants J_1, $C_{1,1}$ and $S_{1,1}$ are all zero and the summation can start at $n = 2$, since the $n = 1$ term contributes nothing; if the z-axis is chosen to be one of the principal axes of inertia of the celestial body, the constants $C_{2,1}$ and $S_{2,1}$ are zero.

The angles α and λ are defined as follows: α is the angle between the position vector of point B and its projection on the x,y plane; λ is the angle between the positive x-axis and this projection. They are the result of the arbitrary choice of the x,y plane as a reference plane.

From this arbitrary choice, there results a singularity: If point B is on the z-axis, λ is not determined and the potential is not defined.

The attractive force is given by

$$\vec{F} = \text{grad } \phi = \nabla\phi = \frac{\partial\phi}{\partial r}\nabla r + \frac{\partial\phi}{\partial(\sin\alpha)}\nabla(\sin\alpha) + \frac{\partial\phi}{\partial\lambda}\nabla\lambda \qquad (2)$$

Here, the problem not only subsists (λ is present in $\frac{\partial\phi}{\partial\lambda}$, $\frac{\partial\phi}{\partial(\sin\alpha)}$, and $\frac{\partial\phi}{\partial\lambda}$) but is even aggravated by the presence of an indeterminate factor. The gradients (see appendix A) are

$$\nabla r = \hat{R}$$

(unit vector along the position vector of point B),

$$\nabla(\sin\alpha) = \frac{1}{r}\hat{k} - \frac{\sin\alpha}{r}\hat{R}$$

($\hat{k}$ is the unit vector along the positive z-axis), and

$$\nabla\lambda = \frac{1}{r(1 - \sin^2\alpha)}\hat{k} \times \hat{R}$$

As $\alpha \rightarrow 90^\circ$, $1 - \sin^2\alpha \rightarrow 0$, $\hat{k} \times \hat{R} \rightarrow \vec{0}$, and the factor $\nabla\lambda$ is indeterminate.

To obtain a representation such that the potential and the force can be unambiguously obtained for points on the z-axis, a different reference plane may be used (see fig. 1).

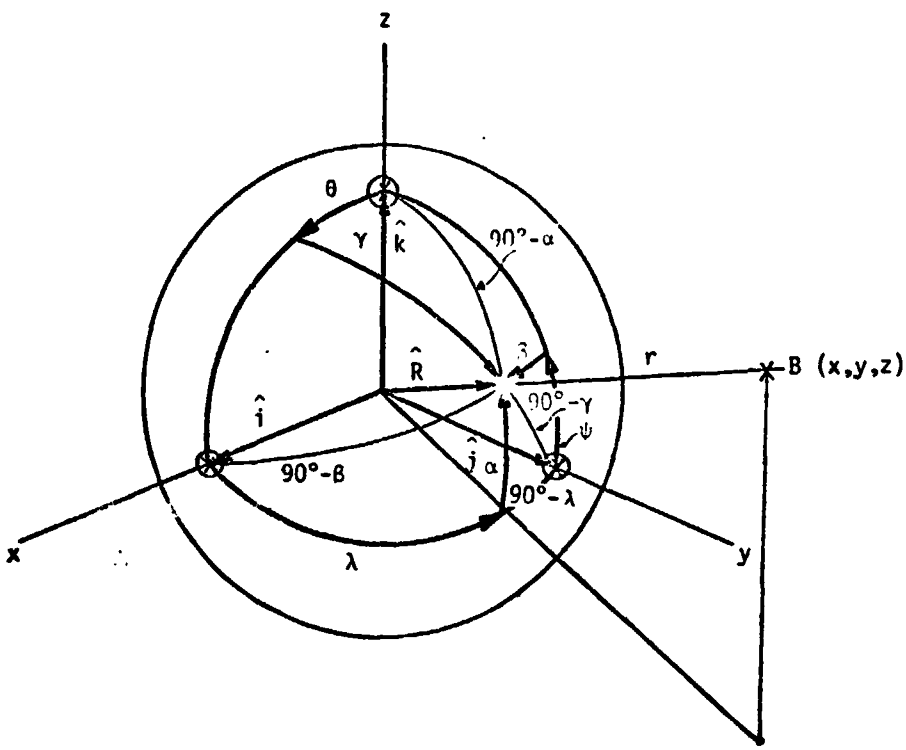

Figure 1. - Coordinates of point B.

But then, the problem has merely been shifted. If the y,z plane were chosen to be the reference plane, the coordinates would be r, β, and ψ; for points on the x-axis, the coordinate ψ would be undefined. If the z,x plane were chosen, the coordinates of the point would be r, γ, and θ; and again, the θ coordinate would be indeterminate for points on the y-axis. Thus for the potential itself; to see what results for the force, consider the gradients of the spherical coordinates in the three cases mentioned.

Plane of reference: x,y $\rightarrow$ $\vec{F} = \frac{\partial\phi}{\partial r}\nabla r + \frac{\partial\phi}{\partial(\sin\alpha)}\nabla(\sin\alpha) + \frac{\partial\phi}{\partial\lambda}\nabla\lambda$

$$\left.\begin{aligned} \nabla r &= \hat{R} \\ \nabla(\sin\alpha) &= \frac{1}{r}\hat{k} - \frac{\sin\alpha}{r}\hat{R} \\ \nabla\lambda &= \frac{1}{r(1-\sin^2\alpha)}\hat{k}\times\hat{R} \end{aligned}\right\} \quad (3)$$

Plane of reference: y,z $\rightarrow$ $\vec{F} = \frac{\partial\phi}{\partial r}\nabla r + \frac{\partial\phi}{\partial(\sin\beta)}\nabla(\sin\beta) + \frac{\partial\phi}{\partial\psi}\nabla\psi$

$$\left.\begin{aligned} \nabla r &= \hat{R} \\ \nabla(\sin\beta) &= \frac{1}{r}\hat{i} - \frac{\sin\beta}{r}\hat{R} \\ \nabla\psi &= \frac{1}{r(1-\sin^2\beta)}\hat{i}\times\hat{R} \end{aligned}\right\} \quad (4)$$

Plane of reference: z,x $\rightarrow$ $\vec{F} = \frac{\partial\phi}{\partial r}\nabla r + \frac{\partial\phi}{\partial(\sin\gamma)}\nabla(\sin\gamma) + \frac{\partial\phi}{\partial\theta}\nabla\theta$

$$\left.\begin{aligned} \nabla r &= \hat{R} \\ \nabla(\sin\gamma) &= \frac{1}{r}\hat{j} - \frac{\sin\gamma}{r}\hat{R} \\ \nabla\theta &= \frac{1}{r(1-\sin^2\gamma)}\hat{j}\times\hat{R} \end{aligned}\right\} \quad (5)$$

The problem subsists: In $\nabla\psi$ and $\nabla\theta$, the same type of indeterminate form that had been found in $\nabla\lambda$ appears again.

3.2 The Proposed Method of Solution

In reference 1, the method chosen to remedy this situation consists of replacing the spherical coordinates r, α, λ (or r, β, ψ or r, γ, θ) with another "distance and direction" representation of a point's position. For this, use is made of the direction cosines, which are always clearly defined for any direction in space. The position of each point is then given by four quantities: the three direction cosines that define the orientation of the position vector and r, the magnitude of that vector.

Call the direction cosines s, t, and u: these are the cosines of the angles between the position vector and the x-axis, the y-axis, and the z-axis, respectively. Therefore,

$$\left.\begin{aligned} s &= \cos(90^\circ - \beta) = \sin\beta \\ t &= \cos(90^\circ - \gamma) = \sin\gamma \\ u &= \cos(90^\circ - \alpha) = \sin\alpha \end{aligned}\right\} \tag{6}$$

On the other hand, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = r(s\hat{i} + t\hat{j} + u\hat{k})$; so alternative expressions for s, t, and u are

$$\left.\begin{aligned} s &= \frac{x}{r} \\ t &= \frac{y}{r} \\ u &= \frac{z}{r} \end{aligned}\right\} \tag{7}$$

and these are the expressions that must be used for numerical work. The attractive force will then be given by

$$\vec{F} = \frac{\partial\phi}{\partial r}\nabla r + \frac{\partial\phi}{\partial(\sin\beta)}\nabla(\sin\beta) + \frac{\partial\phi}{\partial(\sin\gamma)}\nabla(\sin\gamma) + \frac{\partial\phi}{\partial(\sin\alpha)}\nabla(\sin\alpha)$$

By using the expressions for ∇r, $\nabla(\sin\alpha)$, $\nabla(\sin\beta)$, and $\nabla(\sin\gamma)$ found in equations (3), (4), and (5), there results

$$\vec{F} = \frac{\partial\phi}{\partial r}\hat{R} + \frac{\partial\phi}{\partial(\sin\beta)}\left(\frac{1}{r}\hat{i} - \frac{\sin\beta}{r}\hat{R}\right) + \frac{\partial\phi}{\partial(\sin\gamma)}\left(\frac{1}{r}\hat{j} - \frac{\sin\gamma}{r}\hat{R}\right)$$
$$+ \frac{\partial\phi}{\partial(\sin\alpha)}\left(\frac{1}{r}\hat{k} - \frac{\sin\alpha}{r}\hat{R}\right)$$

or by writing s, t, and u for $\sin\beta$, $\sin\gamma$, and $\sin\alpha$, respectively, and collecting terms, there results

$$\vec{F} = \left(\frac{\partial\phi}{\partial r} - \frac{s}{r}\frac{\partial\phi}{\partial s} - \frac{t}{r}\frac{\partial\phi}{\partial t} - \frac{u}{r}\frac{\partial\phi}{\partial u}\right)\hat{R} + \frac{1}{r}\frac{\partial\phi}{\partial s}\hat{i} + \frac{1}{r}\frac{\partial\phi}{\partial t}\hat{j} + \frac{1}{r}\frac{\partial\phi}{\partial u}\hat{k} \tag{8}$$

which is expression (13) of reference 1.

The term in $\nabla\lambda$ has thus been removed from $\vec{F}$. The task has not been completed yet, however; it still remains to express ϕ and $\vec{F}$ in the new variables.

3.3 The "Derived" Legendre Functions

Certain terms in $\vec{F}$ (but not in ϕ) have singularities that stem from another source: The expression for $P_{n,m}(\sin\alpha)$ is, for general n and m (see, for instance, refs. 3 and 4),

$$P_{n,m}(\sin\alpha) = (1 - \sin^2\alpha)^{m/2} \frac{1}{2^n n!} \frac{d^{n+m}}{d(\sin\alpha)^{n+m}} (\sin^2\alpha - 1)^n$$

$$= (1 - \sin^2\alpha)^{m/2} \frac{d^m P_n(\sin\alpha)}{d(\sin\alpha)^m}$$

No problem exists for $m > 1$; but for $m = 1$, the first derivatives $\frac{dP_{n,1}(\sin\alpha)}{d(\sin\alpha)}$ are infinite at $\alpha = \pm 90°$. Write u for $\sin\alpha$ because of equations (6) and differentiate.

$$\frac{dP_{n,1}}{du} = -u(1 - u^2)^{-1/2} \frac{1}{2^n n!} \frac{d^{n+1}}{du^{n+1}} (u^2 - 1)^n + \frac{(1 - u^2)^{1/2}}{2^n n!} \frac{d^{n+2}}{du^{n+2}} (u^2 - 1)^n$$

But $\frac{d^{n+1}}{du^{n+1}} (u^2 - 1)^n$ is a polynomial of degree $n - 1$ in u, which does not become zero when $u = \pm 1$. The singularity, then, comes from the factor $(1 - u^2)^{-1/2}$.

Write

$$A_{n,m}(\sin\alpha) = \frac{1}{2^n n!} \frac{d^{n+m}}{d(\sin\alpha)^{n+m}} (\sin^2\alpha - 1)^n$$

$$= \frac{d^m}{d(\sin\alpha)^m} P_n(\sin\alpha) \tag{9}$$

and note that $(1 - \sin^2\alpha)^{m/2} = \cos^m\alpha$. Then the summation in m in the expression of the potential becomes

$$\sum_{m=1}^{n} A_{n,m}(\sin\alpha)(C_{n,m} \cos^m\alpha \cos m\lambda + S_{n,m} \cos^m\alpha \sin m\lambda)$$

No singularities arise from differentiating $A_{n,m}$ for any value of α. (The $A_{n,m}$ are polynomials.) This was done at the cost of burdening the terms in λ with the $\cos^m\alpha$ coefficient. The $A_{n,m}$ are called by Pines the "derived" Legendre functions.

3.4 The Complex Representation

It remains, then, to express the terms $\cos^m\alpha \cos m\lambda$ and $\cos^m\alpha \sin m\lambda$ in terms of all or some of the variables r, s, t, and u. Reference 1 contains a very ingenious realization of this demand.

Note that these terms appear in a consistent form: The exponent of the $\cos \alpha$ is equal to the coefficient of λ, in all cases. A similar behavior is to be found, by the application of de Moivre's formula, in the powers of a complex number in polar form (see refs. 3 or 4). Consider the complex number $\zeta = \xi + i\eta$, where ξ, η, and ζ are numbers and $i = \sqrt{-1}$, and get, successively,

$$\zeta = \rho e^{i\Phi}$$

in polar form, with $\rho = \sqrt{\xi^2 + \eta^2}$ and $\Phi = \arctan \frac{\eta}{\xi}$,

$$\zeta = \rho(\cos \Phi + i \sin \Phi)$$

by Euler's formula; and

$$\zeta^m = \rho^m(\cos m\Phi + i \sin m\Phi)$$

by de Moivre's formula.

This is a new complex number $w = \zeta^m$, with real and imaginary parts

$$\mathrm{Re}[\zeta^m] = \rho^m \cos m\Phi$$

$$\mathrm{Imag}[\zeta^m] = \rho^m \sin m\Phi$$

A complete analogy is found between the complex number ζ^m and the terms $\cos^m\alpha \cos m\lambda$ and $\cos^m\alpha \sin m\lambda$. These behave as the real and imaginary parts of the m-th power of the complex number

$$\zeta = \cos \alpha e^{i\lambda}$$

$$= \cos \alpha(\cos \lambda + i \sin \lambda) \quad . \tag{10}$$

Therefore, express $\cos \alpha \cos \lambda$ and $\cos \alpha \sin \lambda$ in terms of the variables r, s, t, and u (or some of them), form the complex number $\cos \alpha \cos \lambda + i \cos \alpha \sin \lambda$, and raise the complex number to the appropriate power to obtain the terms needed.

To find $\cos\alpha\cos\lambda$ and $\cos\alpha\sin\lambda$ in the required variables, refer back to figure 1 and notice the spherical triangle with sides λ, α, and $90^\circ - \beta$. The angle opposite to the side $90^\circ - \beta$ is 90°; application of the cosine law (see, for instance, ref. 4) gives

$$\cos(90^\circ - \beta) = \cos\lambda\cos\alpha + \sin\lambda\sin\alpha\cos 90^\circ$$

or

$$\sin\beta = \cos\alpha\cos\lambda \tag{11}$$

Take, now, the spherical triangle with sides α, $90^\circ - \lambda$, and $90^\circ - \gamma$. The angle opposite to the side $90^\circ - \gamma$ is 90°; apply the cosine law and find

$$\cos(90^\circ - \gamma) = \cos\alpha\cos(90^\circ - \lambda) + \sin\alpha\sin(90^\circ - \lambda)\cos 90^\circ$$

or

$$\sin\gamma = \cos\alpha\sin\lambda \tag{12}$$

But $\sin\beta$ and $\sin\lambda$ are the direction cosines s and t, respectively. Then, $s = \cos\alpha\cos\lambda$, $t = \cos\alpha\sin\lambda$, and the complex number wanted is $\zeta = s + it$. In reference 1, the symbols $r_m(s,t)$ and $i_m(s,t)$ are used to represent the real and imaginary parts, respectively, of $w = \zeta^m$.

The representation of ϕ becomes, at this point,

$$\phi = \frac{\mu}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n [J_n A_{n,o}(u) - \sum_{m=1}^{n} A_{n,m}(u)(C_{n,m}r_m(s,t) + S_{n,m}i_m(s,t))]\right\} \tag{13}$$

since, from equation (9), with u written for $\sin\alpha$, $A_{n,m}(u) = \dfrac{d^m P_n(u)}{du^m}$ becomes $A_{n,o}(u) = P_n(u)$ when $m = 0$.

There are no singularities arising from the $A_{n,m}(u)$, which are the only terms in u. There are none from the terms in r. (The only one would be for $r = 0$, but the expression of the potential is only valid outside a sphere of radius a and center at the origin.) The terms $r_m(s,t)$ and $i_m(s,t)$ and their derivatives with respect to s and t can be obtained from $w = \zeta^m$. Since this is an analytic

function of ζ for all the values of m considered (integer and nonnegative), the derivatives of all orders exist and are continuous. Differentiate $w = \zeta^m$ with respect to ζ.

$$\frac{dw}{d\zeta} = m\zeta^{m-1}$$
$$= m[r_{m-1}(s,t) + i i_{m-1}(s,t)]$$

But

$$\frac{dw}{d\zeta} = \frac{\partial r_m}{\partial s} + i\,\frac{\partial i_m}{\partial s}$$
$$= \frac{\partial i_m}{\partial t} - i\,\frac{\partial r_m}{\partial t}$$

from the Cauchy-Riemann conditions (see appendix B), which must be satisfied because of the fact that $w = \zeta^m$ is analytic. Since equality of complex numbers implies separate equalities for real and imaginary parts, the following expressions are obtained.

$$\left.\begin{aligned} \frac{\partial r_m}{\partial s} &= \frac{\partial i_m}{\partial t} = m r_{m-1} \\ \frac{\partial r_m}{\partial t} &= -\frac{\partial i_m}{\partial s} = m i_{m-1} \end{aligned}\right\} \qquad (14)$$

These expressions give the very valuable information that the derivatives of the m-th terms may be obtained with no need to perform any differentiation, but only to multiply by m the m-1st terms.

3.5 The Recurrence Process

The terms in r, in u, and in (s,t) may all be obtained recursively. The same is true of their derivatives of all orders.

Recursive processes are ideally suited for use in computers.

The next subsections are devoted to the derivation of the recursion formulas for the various functions involved.

This is one of the most important parts of this note; it is upon the recursive process that the efficiency of the formulation depends.

3.5.1 Complex variable recursion.- The terms required, r_m and i_m, are the real and imaginary parts of $w = \zeta^m$. To obtain them recursively, form ζ^m from ζ^{m-1}.

$$\begin{aligned}\zeta^m &= r_m + i i_m \\ &= \zeta^{m-1}\zeta \\ &= (r_{m-1} + i i_{m-1})(s + it) \\ &= (sr_{m-1} - ti_{m-1}) + i(si_{m-1} + tr_{m-1})\end{aligned}$$

Therefore,

$$\left.\begin{aligned} r_m &= sr_{m-1} - ti_{m-1} \\ i_m &= si_{m-1} + tr_{m-1}\end{aligned}\right\} \tag{15}$$

These are equations (25) of reference 1. Besides these, there are two others, also under the number (25), but they are wrong (see appendix C).

The derivatives of r_m and i_m are given by equations (14) of this note.

To start the recurrence process, two values are needed. Since $\zeta = s + it$,

$$r_1 = s$$

$$i_1 = t$$

Equations (15) are valid for all integer values of m. The recursion could have been started with any other known value, such as $\zeta^o = 1$, which would give $r_o = 1$, $i_o = 0$.

The representation of the potential, equation (13), requires $C_{n,m}r_m + S_{n,m}i_m$. The process of generation of the powers ζ^m can be used for this and its derivatives. Define

$$D_{n,m} = C_{n,m}r_m + S_{n,m}i_m \tag{16}$$

Differentiate with respect to s and to t.

$$\frac{\partial D_{n,m}}{\partial s} = C_{n,m}\frac{\partial r_m}{\partial s} + S_{n,m}\frac{\partial i_m}{\partial s}$$

$$= m(C_{n,m}r_{m-1} + S_{n,m}i_{m-1})$$

$$\frac{\partial D_{n,m}}{\partial t} = C_{n,m}\frac{\partial r_m}{\partial t} + S_{n,m}\frac{\partial i_m}{\partial t}$$

$$= m(S_{n,m}r_{m-1} - C_{n,m}i_{m-1})$$

Now define

$$E_{n,m} = C_{n,m}r_{m-1} + S_{n,m}i_{m-1}$$

$$F_{n,m} = S_{n,m}r_{m-1} - C_{n,m}i_{m-1}$$

Therefore,

$$\left.\begin{aligned} \frac{\partial D_{n,m}}{\partial s} &= mE_{n,m} \\ \frac{\partial D_{n,m}}{\partial t} &= mF_{n,m} \end{aligned}\right\} \tag{17}$$

This is all that is needed to find $\vec{F}$ from ϕ. But second derivatives are needed for the variational equations. Therefore, continue the process by differentiating again $D_{n,m}$ with respect to s and t and defining new functions $G_{n,m}$ and $H_{n,m}$; then find the relations

$$\left.\begin{aligned} \frac{\partial E_{n,m}}{\partial s} &= -\frac{\partial F_{n,m}}{\partial t} = (m-1)G_{n,m} \\ \frac{\partial E_{n,m}}{\partial t} &= \frac{\partial F_{n,m}}{\partial s} = (m-1)H_{n,m} \end{aligned}\right\} \tag{18}$$

Also, there results

$$\frac{\partial^2 D_{n,m}}{\partial s^2} + \frac{\partial^2 D_{n,m}}{\partial t^2} = 0$$

($D_{n,m}$ are harmonic functions) and

$$\frac{\partial^2 D_{n,m}}{\partial s \partial t} = \frac{\partial^2 D_{n,m}}{\partial t \partial s}$$

as was to be expected because of the continuity of derivatives (of all orders) of ζ.

The functions $G_{n,m}$ and $H_{n,m}$ are

$$G_{n,m} = C_{n,m} r_{m-2} + S_{n,m} i_{m-2}$$

$$H_{n,m} = S_{n,m} r_{m-2} - C_{n,m} i_{m-2}$$

3.5.2 <u>"Derived" Legendre functions recursion</u>.- The $A_{n,m}(u)$ are derivatives of the $P_n(u)$.

$$P_n(u) = \frac{1}{2^n n!} \frac{d^n}{du^n} (u^2 - 1)^n$$

by Rodrigues' formula (see refs. 3 or 4) and

$$A_{n,m}(u) = \frac{1}{2^n n!} \frac{d^{n+m}}{du^{n+m}} (u^2 - 1)^n$$

by definition; so

$$A_{n,m} = \frac{d^m}{du^m} P_n$$

with $A_{n,0} = P_n$.

The Legendre polynomials satisfy various recurrence relations (see, for instance, ref. 4).

$$\left.\begin{aligned} &\text{a)}\quad (n+1)P_{n+1}(u) - (2n+1)uP_n(u) + nP_{n-1}(u) = 0 \\ &\text{b)}\quad \frac{d}{du}[P_{n+1}(u)] - u\frac{d}{du}[P_n(u)] = (n+1)P_n(u) \\ &\text{c)}\quad u\frac{d}{du}[P_n(u)] - \frac{d}{du}[P_{n-1}(u)] = nP_n(u) \end{aligned}\right\} \qquad (19)$$

$$\left.\begin{aligned} &\text{d)}\ \frac{d}{du}[P_{n+1}(u)] - \frac{d}{du}[P_{n-1}(u)] = (2n+1)P_n(u) \\ &\text{e)}\ (u^2-1)\frac{d}{du}[P_n(u)] = nuP_n(u) - nP_{n-1}(u) \end{aligned}\right| \tag{19}$$

In terms of the $A_{n,m}(u)$, these relations would be written as

$$\left.\begin{aligned} &\text{a}_1)\ (n+1)A_{n+1,0}(u) - (2n+1)uA_{n,0}(u) + nA_{n-1,0}(u) = 0 \\ &\text{b}_1)\ A_{n+1,1}(u) - uA_{n,1}(u) = (n+1)A_{n,0}(u) \\ &\text{c}_1)\ uA_{n,1}(u) - A_{n-1,1}(u) = nA_{n,0}(u) \\ &\text{d}_1)\ A_{n+1,1}(u) - A_{n-1,1}(u) = (2n+1)A_{n,0}(u) \\ &\text{e}_1)\ (u^2-1)A_{n,1}(u) = nuA_{n,0}(u) - nA_{n-1,0}(u) \end{aligned}\right\} \tag{20}$$

If these relations are differentiated with respect to u, an m number of times, the following are obtained.

$$\left.\begin{aligned} &\text{a}_2)\ (n+1)A_{n+1,m} = m(2n+1)A_{n,m-1} + (2n+1)uA_{n,m} + nA_{n-1,m} \\ &\text{b}_2)\ A_{n+1,m+1} = (n+m+1)A_{n,m} + uA_{n,m+1} \\ &\text{c}_2)\ (n-m)A_{n,m} = uA_{n,m+1} - A_{n-1,m+1} \\ &\text{d}_2)\ A_{n+1,m+1} = (2n+1)A_{n,m} + A_{n-1,m+1} \\ &\text{e}_2)\ m(n-m+1)A_{n,m-1} = (u^2-1)A_{n,m+1} + (2m-n)uA_{n,m} + nA_{n-1,m} \end{aligned}\right\} \tag{21}$$

The following information is available: For $m > n$, $A_{n,m}(u) = 0$; for $m = n$, $A_{n,n}(u) = \frac{(2n)!}{2^n n!}$; and for $m = n - 1$, $A_{n,n-1}(u) = uA_{n,n}(u)$. These are all straightforward consequences of the definition of $A_{n,m}$; each $A_{n,m}$ is a polynomial of degree $n - m$ with only odd or only even powers of u, according to whether $n - m$ is odd or even.

Form a table of the $A_{n,m}$, with consideration given all $\underline{n}$ and $\underline{m}$ from zero on up. Let $\underline{n}$ indicate the row and $\underline{m}$ the column of the matrix thus formed. All the elements of the diagonal are found by means of the expression

$A_{n,n}(u) = 1 \cdot 3 \cdot 5 \cdot 7 \; . \; . \; . \; . \; . \; (2n - 1) = \frac{(2n)!}{2^n n!}$; all elements of the upper triangular matrix are zero; and the first column has the Legendre polynomials. Each element immediately to the left of an element of the main diagonal is the same, multiplied by $\underline{u}$.

It remains to fill the empty spaces. This can be done by any of the schemes that constitute the recursion relations a_2 through e_2.

Construct the matrix, fill in the known spaces, and try to see how the various schemes work. A rectangle indicates the element derived; circles indicate those it is derived from (see table I).

TABLE I. - RECURSION SCHEMES FOR THE $A_{n,m}$

n	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	u	1	0	0	0	0	0	0	0
2	$\boxed{\frac{1}{2}(3u^2-1)}$	3u	3	0	0	0	0	0	0
3	$\frac{1}{2}(5u^3-3u)$		15u	15	0	0	0	0	0
4	$\frac{1}{8}(35u^4-30u^2+3)$			105u	105	0	0	0	0
5	$\frac{1}{8}(63u^5-70u^3+15u)$				945u	945	0	0	0
6	$\frac{1}{16}(231u^6-315u^4+105u^2-5)$					10395u	10395	0	0
7	$\frac{1}{16}(429u^7-693u^5+315u^3-35u)$						135135u	135135	0
8	$\frac{1}{128}(6435u^8-12012u^6+6930u^4-1260u^2+35)$							2027025u	2027025

a_2) For $m = 0$, obtain each element of the first column in terms of the two above – indicated by solid lines around the elements; for $m \neq 0$, find each element (not of the first column) in terms of three others: two immediately above and one on the previous column – indicated by dotted lines (. . . .).

b_2) Obtain elements (not of the first column) in terms of two others of the row above – indicated by dashed lines (- - - -).

c_2) Get elements (not of the diagonal) in terms of two others of the following column – indicated by two dots-dash (..-..-..-).

d_2) Find elements (not of the first column) in terms of two others (both of previous rows), one of the same column and another of the previous – indicated by dash-one dot (-.-.-.-).

e_2) Get elements (not of the $n = 2$ row) in terms of three others, two of the same row and following columns and the other of the previous row and following column – indicated by one dot-two dashes (—.——.——.——).

In reference 1 (where only relations b_2 and c_2 are used), relation b_2 is used to simplify the derivatives of the potential. whereas c_2 is used for the actual recursive determination of the $A_{n,m}$. The reasoning behind this last choice is that c_2 is a more stable formula than b_2.

Consider the same term calculated from both formulas: from b_2,
$A_{n+1,m+1} = (n + m + 1)A_{n,m} + uA_{n,m+1}$; from c_2,
$A_{n+1,m+1} = \frac{1}{n - m}(uA_{n+1,m+2} - A_{n,m+2})$. In b_2, any error in the calculation of $A_{n,m}$ will be multiplied by $(n + m + 1)$ and will therefore carry a larger error into $A_{n+1,m+1}$. These terms are obtained from two terms, one directly above and the other just to the left of it; so as new terms are calculated, the values of n and m are increasing and the errors become progressively larger. If the term is calculated by c_2, any error is divided by $(n - m)$. Terms in the diagonal, where $n - m = 0$, are not calculated this way but serve as starting values. The recursion progresses to the left; and the farther away the term is from the diagonal, the greater the difference between $\underline{n}$ and $\underline{m}$ and the smaller the effect of any initial error. For this reason, relation $_2$ is to be preferred to b_2 for recursion.[1]

Now that the $A_{n,m}$ have been found, the derivatives $\frac{dA_{n,m}}{du}$ must be obtained. From the definition of $A_{n,m}$, it follows that

$$\frac{dA_{n,m}}{du} = A_{n,m+1} \tag{22}$$

[1]The same argument holds without change in comparisons between c_2 and d_2 or a_2. In e_2, the effects of errors are decreased as in c_2; but since each term is derived from three others, there are more sources of error.

The derivative of $A_{n,m}$ may then be obtained by using the recursion relations, with no need for an actual differentiation to be performed.

3.5.3 Recursion for terms in the radial distance.- Another quantity that may be obtained recursively is the term containing r. Actually, this will lead to a more compact representation of the potential.

The potential was given by

$$\phi = \frac{\mu}{r}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{a}{r}\right)^n [J_n A_{n,o}(u) - \sum_{m=1}^{n} A_{n,m}(u)D_{n,m}(s,t)]\right\}$$

$$= \frac{\mu}{r} - \sum_{n=1}^{\infty} \frac{\mu}{r}\left(\frac{a}{r}\right)^n [J_n A_{n,o}(u) - \sum_{m=1}^{n} A_{n,m}(u)D_{n,m}(s,t)]$$

obtained by using equation (16) in equation (13).

The lowest $D_{n,m}$ defined is $D_{1,1}$. Let $m = 0$ in $D_{n,m}$ and get $D_{n,o} = C_{n,o}r_o(s,t) + S_{n,o}i_o(s,t) = C_{n,o}$ because, as had been seen, $r_o = 1$ and $i_o = 0$.

So if $C_{n,o}$ are defined as $-J_n$ for $n = 0$, the following simpler expression for the potential is found.

$$\phi = \frac{\mu}{r} + \sum_{n=1}^{\infty} \frac{\mu}{r}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} A_{n,m}(u)D_{n,m}(s,t) \tag{23}$$

A better form may be obtained by allowing $n = 0$ as a possible value. This entails defining a $D_{o,o} = 1$ and $A_{o,o} = 1$. In fact, $A_{o,o} = 1$ had already appeared; so only the $D_{o,o}$ remains to be defined.

Since $D_{o,o} = C_{o,o}$, this is, in the final analysis, the only definition needed: $C_{o,o} = 1$.

The expression for the potential becomes

$$\phi = \sum_{n=0}^{\infty} \frac{\mu}{r}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} A_{n,m}(u)D_{n,m}(s,t) \tag{24}$$

It is now a simple matter to find a recursive representation for the term in r. Let $\rho_n = \frac{\mu}{r}\left(\frac{a}{r}\right)^n$. When $n = 0$, $\rho_o = \frac{\mu}{r}$; when $n > 0$, $\rho_n = \frac{a}{r}\rho_{n-1}$. Let $\frac{a}{r} = \rho$, and it follows that $\rho_n = \rho\rho_{n-1}$.

As for the derivatives,

$$\rho_o = \frac{\mu}{r} = \frac{r}{a}\rho_1 \rightarrow \frac{d\rho_o}{dr} = -\frac{\mu}{r^2} = -\frac{1}{a}\rho_1$$

$$\rho_1 = \frac{\mu a}{r^2} = \frac{r}{a}\rho_2 \rightarrow \frac{d\rho_1}{dr} = -\frac{2\mu a}{r^3} = -\frac{2}{a}\rho_2$$

$$\rho_2 = \frac{\mu a^2}{r^3} = \frac{r}{a}\rho_3 \rightarrow \frac{d\rho_2}{dr} = -\frac{3\mu a^2}{r^4} = -\frac{3}{a}\rho_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\rho_n = \rho\rho_{n-1} = \frac{r}{a}\rho_{n+1} \rightarrow \frac{d\rho_n}{dr} = -\frac{n+1}{a}\rho_{n+1} \tag{25}$$

Note the relation

$$\frac{\rho_n}{r} = \frac{\rho_{n+1}}{a} \tag{26}$$

3.5.4 Collection of formulas.- The potential can then be written as

$$\phi = \sum_{n=0}^{\infty} \rho_n \sum_{m=0}^{n} A_{n,m} D_{n,m} \tag{27}$$

which is formula (29) of reference 1.

The expression for the force was equation (7).

$$\vec{F} = \frac{1}{r}\frac{\partial\phi}{\partial s}\hat{i} + \frac{1}{r}\frac{\partial\phi}{\partial t}\hat{j} + \frac{1}{r}\frac{\partial\phi}{\partial u}\hat{k} + \left(\frac{\partial\phi}{\partial r} - \frac{s}{r}\frac{\partial\phi}{\partial s} - \frac{t}{r}\frac{\partial\phi}{\partial t} - \frac{u}{r}\frac{\partial\phi}{\partial u}\right)\hat{R}$$

or, in the notation of reference 1,

$$\vec{F} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} + a_4\hat{R} \tag{28}$$

The expressions for a_1, a_2, a_3, and a_4 can now be found from the differentiation recursions found in subsections 3.5.1, 3.5.2, and 3.5.3.

$$\begin{aligned} a_1 &= \frac{1}{r}\frac{\partial\phi}{\partial s} \\ &= \frac{1}{r}\frac{\partial}{\partial s}\left(\sum_{n=0} \rho_n \sum_{m=0}^{n} A_{n,m}D_{n,m}\right) \\ &= \sum_{n=0} \frac{\rho_n}{r} \sum_{m=0}^{n} A_{n,m}\frac{\partial D_{n,m}}{\partial s} \\ &= \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}E_{n,m} \end{aligned} \tag{29}$$

Similarly,

$$a_2 = \frac{1}{r}\frac{\partial\phi}{\partial t}$$

$$= \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}F_{n,m} \tag{30}$$

$$a_3 = \frac{1}{r}\frac{\partial\phi}{\partial u}$$

$$= \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} \frac{\partial A_{n,m}}{\partial u} D_{n,m}$$

$$= \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m+1}D_{n,m} \tag{31}$$

$$a_4 = \frac{\partial\phi}{\partial r} - sa_1 - ta_2 - ua_3$$

But

$$\frac{\partial\phi}{\partial r} = \sum_{n=0}^{\infty} \frac{d\rho_n}{dr} \sum_{m=0}^{n} A_{n,m}D_{n,m} = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} (n+1)A_{n,m}D_{n,m}$$

Now

$$-ua_3 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} uA_{n,m+1}D_{n,m}$$

and

$$-sa_1 - ta_2 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}(sE_{n,m} + tF_{n,m})$$

$$= -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}\,[C_{n,m}(sr_{m-1} - ti_{m-1})$$

$$+ S_{n,m}(si_{m-1} + tr_{m-1})]$$

$$= -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}D_{n,m}$$

So, by collecting all these,

$$a_4 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} [(n + m + 1)A_{n,m} + uA_{n,m+1}]D_{n,m}$$

And with the help of relation b_2 from equations (21),

$$a_4 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n+1,m+1}D_{n,m} \tag{32}$$

$\vec{F}$ can thus be written as

$$\vec{F} = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} [mA_{n,m}(E_{n,m}\hat{i} + F_{n,m}\hat{j}) + A_{n,m+1}D_{n,m}\hat{k}$$

$$-A_{n+1,m+1}D_{n,m}\hat{R}] \tag{33}$$

Another way to write $\vec{F}$, taking into account the fact that $\hat{R} = s\hat{i} + t\hat{j} + u\hat{k}$, is

$$\vec{F} = (a_1 + a_4 s)\hat{i} + (a_2 + a_4 t)\hat{j} + (a_3 + a_4 u)\hat{k} \tag{34}$$

The main formulas are now presented together.

Formulas for the variables:

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$\left.\begin{aligned} s &= \frac{x}{r} \\ t &= \frac{y}{r} \\ u &= \frac{z}{r} \end{aligned}\right\} \tag{7}$$

For the recursions:

$$\rho = \frac{a}{r}$$

$$\rho_o = \frac{\mu}{r}$$

$$\rho_n = \rho\rho_{n-1} \tag{25}$$

$$A_{n,n} = \frac{(2n)!}{2^n n!}$$

$$A_{n,n-1} = uA_{n,n}$$

$$A_{n,m} = \frac{1}{n - m}(uA_{n,m+1} - A_{n-1,m+1}) \tag{21}$$

$$r_o = 1$$

$$i_o = 0$$

$$r_m = sr_{m-i} - ti_{m-1}$$

$$i_m = si_{m-1} + tr_{m-1} \tag{15}$$

$$E_{n,m} = C_{n,m}r_{m-1} + S_{n,m}i_{m-1}$$

$$F_{n,m} = S_{n,m}r_{m-1} - C_{n,m}i_{m-1} \tag{16}$$

$$D_{n,m} = C_{n,m}r_m + S_{n,m}i_m$$

For the potential:

$$\phi = \sum_{n=0}^{\infty} \rho_n \sum_{m=0}^{n} A_{n,m}D_{n,m} \tag{27}$$

For the attractive force:

$$a_1 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}E_{n,m} \tag{29}$$

$$a_2 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}F_{n,m} \tag{30}$$

$$a_3 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m+1}D_{n,m} \tag{31}$$

$$a_4 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n+1,m+1}D_{n,m} \tag{32}$$

$$\vec{F} = (a_1 + a_4 s)\hat{i} + (a_2 + a_4 t)\hat{j} + (a_3 + a_4 u)\hat{k} \tag{34}$$

3.6 The Second Derivatives

In this section, the second-order partial derivatives of ϕ are found. This can be done in several ways. In reference 1, the following way is chosen.

Consider the vector

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

The "gradient of the vector" is defined to be the sum of outer products.

$$\frac{\partial^2\phi}{\partial\vec{R}^2} = \frac{\partial\vec{F}}{\partial\vec{R}} = \nabla(F_x)\hat{i}^T + \nabla(F_y)\hat{j}^T + \nabla(F_z)\hat{k}^T$$

This is a 3 x 3 matrix, P. The vector $\vec{F}$, in this case, was given by equation (34).

$$\vec{F} = (a_1 + a_4 s)\hat{i} + (a_2 + a_4 t)\hat{j} + (a_3 + a_4 u)\hat{k}$$

The matrix is found to be

$$P = \frac{\partial^2\phi}{\partial\vec{R}^2} = \begin{bmatrix} \frac{\partial}{\partial x}(a_1 + a_4 s) & \frac{\partial}{\partial x}(a_2 + a_4 t) & \frac{\partial}{\partial x}(a_3 + a_4 u) \\ \frac{\partial}{\partial y}(a_1 + a_4 s) & \frac{\partial}{\partial y}(a_2 + a_4 t) & \frac{\partial}{\partial y}(a_3 + a_4 u) \\ \frac{\partial}{\partial z}(a_1 + a_4 s) & \frac{\partial}{\partial z}(a_2 + a_4 t) & \frac{\partial}{\partial z}(a_3 + a_4 u) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \tag{35}$$

and it may be seen that it is symmetric.

T - various elements of this matrix may be found by applying to the functions in parentheses the same procedure that was applied to the function ϕ. The expression for $\nabla\phi$ was

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

the components being given by

$$\frac{\partial\phi}{\partial x} = a_1 + a_4 s = \frac{1}{r}\frac{\partial\phi}{\partial s} + s\left(\frac{\partial\phi}{\partial r} - \frac{s}{r}\frac{\partial\phi}{\partial s} - \frac{t}{r}\frac{\partial\phi}{\partial t} - \frac{u}{r}\frac{\partial\phi}{\partial u}\right)$$

$$\frac{\partial\phi}{\partial y} = a_2 + a_4 t = \frac{1}{r}\frac{\partial\phi}{\partial t} + t\left(\frac{\partial\phi}{\partial r} - \frac{s}{r}\frac{\partial\phi}{\partial s} - \frac{t}{r}\frac{\partial\phi}{\partial t} - \frac{u}{r}\frac{\partial\phi}{\partial u}\right)$$

$$\frac{\partial\phi}{\partial z} = a_3 + a_4 u = \frac{1}{r}\frac{\partial\phi}{\partial u} + u\left(\frac{\partial\phi}{\partial r} - \frac{s}{r}\frac{\partial\phi}{\partial s} - \frac{t}{r}\frac{\partial\phi}{\partial t} - \frac{u}{r}\frac{\partial\phi}{\partial u}\right)$$

Since $\frac{\partial}{\partial x}(a_1 + a_4 s) = \frac{\partial a_1}{\partial x} + s\frac{\partial a_4}{\partial x} + a_4\frac{\partial s}{\partial x}$, and similarly for other terms, it will be necessary to find $\frac{\partial a_1}{\partial x}$, $\frac{\partial a_1}{\partial y}$, $\frac{\partial a_1}{\partial z}$, $\frac{\partial a_4}{\partial x}$, $\frac{\partial a_4}{\partial y}$, $\frac{\partial a_4}{\partial z}$, $\frac{\partial a_2}{\partial y}$, $\frac{\partial a_2}{\partial z}$, and $\frac{\partial a_3}{\partial z}$. (Because the matrix is symmetric, no others are needed.) As for $\frac{\partial s}{\partial x}$, $\frac{\partial s}{\partial y}$, $\frac{\partial s}{\partial z}$, $\frac{\partial t}{\partial y}$, $\frac{\partial t}{\partial z}$, and $\frac{\partial u}{\partial z}$, recall that $s = \frac{x}{r}$, $t = \frac{y}{r}$, and $u = \frac{z}{r}$ and that s, t, u, and r are independent of one another. Therefore,

$$\frac{\partial s}{\partial x} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial z} = \frac{1}{r}$$

and

$$\frac{\partial s}{\partial y} = \frac{\partial s}{\partial z} = \frac{\partial t}{\partial z} = 0$$

Then

$$\frac{\partial a_1}{\partial x} = \frac{1}{r}\frac{\partial a_1}{\partial s} + s\left(\frac{\partial a_1}{\partial r} - \frac{s}{r}\frac{\partial a_1}{\partial s} - \frac{t}{r}\frac{\partial a_1}{\partial t} - \frac{u}{r}\frac{\partial a_1}{\partial u}\right)$$

and in a similar way for the other derivatives.

Now use the notation, for any function L,

$$\frac{1}{r}\frac{\partial L}{\partial s} = L_1$$

$$\frac{1}{r}\frac{\partial L}{\partial t} = L_2 \qquad (36)$$

$$\frac{1}{r}\frac{\partial L}{\partial u} = L_3$$

$$\frac{\partial L}{\partial r} - \frac{s}{r}\frac{\partial L}{\partial s} - \frac{t}{r}\frac{\partial L}{\partial t} - \frac{u}{r}\frac{\partial L}{\partial u} = L_4$$

and there results that

$$\frac{\partial a_1}{\partial x} = a_{11} + sa_{14}$$

$$\frac{\partial a_1}{\partial y} = a_{12} + ta_{14}$$

$$\frac{\partial a_1}{\partial z} = a_{13} + ua_{14}$$

$$\frac{\partial a_2}{\partial y} = a_{22} + ta_{24}$$

$$\frac{\partial a_2}{\partial z} = a_{23} + ua_{24}$$

$$\frac{\partial a_3}{\partial z} = a_{33} + ua_{34}$$

$$\frac{\partial a_4}{\partial x} = a_{41} + sa_{44}$$

$$\frac{\partial a_4}{\partial y} = a_{42} + ta_{44}$$

$$\frac{\partial a_4}{\partial z} = a_{43} + ua_{44}$$

Note, now, that $a_{ij} = a_{ji}$ for all i and j from 1 through 4 and that $a_{11} = -a_{22}$.

To find P_{11}, add $\frac{\partial a_1}{\partial x}$, $s\,\frac{\partial a_4}{\partial x}$, and $a_4\,\frac{\partial s}{\partial x}$ and get

$$
\begin{aligned}
P_{11} &= a_{11} + sa_{14} + s(a_{41} + sa_{44}) + a_4\,\frac{1}{r} \\
&= a_{11} + 2sa_{14} + s^2 a_{44} + \frac{a_4}{r}
\end{aligned}
$$

Similarly,

$$
\left.
\begin{aligned}
P_{12} &= P_{21} = a_{12} + ta_{14} + sa_{24} + sta_{44} \\
P_{13} &= P_{31} = a_{13} + ua_{14} + sa_{24} + sua_{44} \\
P_{22} &= -a_{11} + 2ta_{24} + t^2 a_{44} + \frac{a_4}{r} \\
P_{23} &= P_{32} = a_{23} + ua_{24} + ta_{34} + uta_{44} \\
P_{33} &= a_{33} + 2ua_{34} + u^2 a_{44} + \frac{a_4}{r}
\end{aligned}
\right\} \qquad (37)
$$

The derivatives a_{11} through a_{44} may be found by the use of formulas (17), (18), (22), and (25).

$$\left.\begin{aligned}
a_{11} &= -a_{22} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} m(m-1)A_{n,m}G_{n,m} \\
a_{12} &= a_{21} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} m(m-1)A_{n,m}H_{n,m} \\
a_{13} &= a_{31} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n,m+1}E_{n,m} \\
a_{14} &= a_{41} = -\sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n+1,m+1}E_{n,m} \\
a_{23} &= a_{32} = \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n,m+1}F_{n,m} \\
a_{24} &= a_{42} = -\sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} mA_{n+1,m+1}F_{n,m} \\
a_{33} &= \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} A_{n,m+2}D_{n,m} \\
a_{34} &= a_{43} = -\sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} A_{n+1,m+2}D_{n,m} \\
a_{44} &= \sum_{n=0}^{\infty} \frac{\rho_{n+2}}{a^2} \sum_{m=0}^{n} A_{n+2,m+2}D_{n,m}
\end{aligned}\right\} \tag{38}$$

3.7 The Variational Equations

A decision on what order terms to include in the potential must be made after studying the effects on the attractive force of changes in the coefficients of the spherical harmonics. For this purpose, the partial derivatives of the force with respect to those coefficients must be found.

For a particular coefficient $C_{N,M}$, the derivative is

$$
\begin{aligned}
\frac{\partial \vec{F}}{\partial C_{N,M}} &= \frac{\partial a_1}{\partial C_{N,M}}\hat{i} + \frac{\partial a_2}{\partial C_{N,M}}\hat{j} + \frac{\partial a_3}{\partial C_{N,M}}\hat{k} + \frac{\partial a_4}{\partial C_{N,M}}\hat{R} \\
&= \left(\frac{\partial a_1}{\partial C_{N,M}} + s\frac{\partial a_4}{\partial C_{N,M}}\right)\hat{i} + \left(\frac{\partial a_2}{\partial C_{N,M}} + t\frac{\partial a_4}{\partial C_{N,M}}\right)\hat{j} + \left(\frac{\partial a_3}{\partial C_{N,M}} + u\frac{\partial a_4}{\partial C_{N,M}}\right)\hat{k} \\
&= f_{C_{N,M,1}}\hat{i} + f_{C_{N,M,2}}\hat{j} + f_{C_{N,M,3}}\hat{k}
\end{aligned}
\tag{39}
$$

The expressions of a_1, a_2, a_3, and a_4 are given by equations (29), (30), (31), and (32).

The derivatives, then, are

$$
\left.
\begin{aligned}
f_{C_{N,M,1}} &= \frac{\rho_{N+1}}{a}\left(MA_{N,M}r_{M-1} - sA_{N+1,M+1}r_M\right) \\
f_{C_{N,M,2}} &= -\frac{\rho_{N+1}}{a}\left(MA_{N,M}i_{M-1} + tA_{N+1,M+1}r_M\right) \\
f_{C_{N,M,3}} &= \frac{\rho_{N+1}}{a}\left(A_{N,M+1}r_M - uA_{N+1,M+1}r_M\right) \\
f_{S_{N,M,1}} &= \frac{\rho_{N+1}}{a}\left(MA_{N,M}i_{M-1} - sA_{N+1,M+1}i_M\right) \\
f_{S_{N,M,2}} &= \frac{\rho_{N+1}}{a}\left(MA_{N,M}r_{M-1} - tA_{N+1,M+1}i_M\right) \\
f_{S_{N,M,3}} &= \frac{\rho_{N+1}}{a}\left(A_{N,M+1}i_M - uA_{N+1,M+1}i_M\right)
\end{aligned}
\right\}
\tag{40}
$$

There are several mistakes in the expressions for these derivatives in reference 1. For a more complete derivation of these results, see appendix C.

Variational equations will be required for several parameters, particularly for the initial conditions.

The equations of motion in an inertial system are obtained by means of a rotation matrix, from the equations relative to a rotating coordinate system.

Let the position vector of the point where the potential is being evaluated be represented by $\vec{R}_B$ in a system of axes fixed to the celestial body and by $\vec{R}_{IN}$ in an inertial system. If the rotation matrix is N, the two representations are connected by the equation

$$\vec{R}_B = N\vec{R}_{IN} \tag{41}$$

The force, which was given by $\vec{F}_B = \nabla_B \phi$ (where ∇_B means gradient with respect to the body-fixed axes), is given in the inertial system by

$$\vec{F}_{IN} = N^T \vec{F}_B \tag{42}$$

Therefore, the equations of motion of the point in the inertial system are

$$\ddot{\vec{R}}_{IN} = N^T \vec{F}_B \tag{43}$$

Consider a parameter Γ_i. Let it be such that second-order differentiation with respect to Γ_i and to t can be performed in any order with the same results; that is,

$$\frac{d}{dt}\left(\frac{\partial \vec{R}_{IN}}{\partial \Gamma_i}\right) = \frac{\partial}{\partial \Gamma_i}\left(\dot{\vec{R}}_{IN}\right) \tag{44}$$

where a dot above the function stands for differentiation with respect to time.

Then differentiate equation (43) with respect to Γ_i and find

$$\frac{d}{dt}\left(\frac{\partial \dot{\vec{R}}_{IN}}{\partial \Gamma_i}\right) = \frac{\partial}{\partial \Gamma_i}\left(\ddot{\vec{R}}_{IN}\right) = \frac{\partial}{\partial \Gamma_i}\left(N^T \vec{F}_B\right) = \frac{\partial N^T}{\partial \Gamma i}\vec{F}_B + N^T \frac{\partial \vec{F}_B}{\partial \Gamma_i} \tag{45}$$

The matrix N^T depends only on the time (duration of the rotation); Γ_i represents a parameter such as the initial conditions or the coefficients of the spherical harmonics, but not the time. Therefore,

$$\frac{\partial N^T}{\partial \Gamma_i} = 0 \tag{46}$$

Then

$$\frac{d}{dt}\left(\frac{\partial \dot{\vec{R}}_{IN}}{\partial \Gamma_i}\right) = N^T \frac{\partial \vec{F}_B}{\partial \Gamma_i}$$

Carry out this last differentiation through $\vec{R}_B$.

$$\frac{d}{dt}\left(\frac{\partial \dot{\vec{R}}_{IN}}{\partial \Gamma_i}\right) = N^T \frac{\partial \vec{F}_B}{\partial \vec{R}_B} \frac{\partial \vec{R}_B}{\partial \Gamma_i} = N^T P \frac{\partial (N\vec{R}_{IN})}{\partial \Gamma_i} = N^T PN \frac{\partial \vec{R}_{IN}}{\partial \Gamma_i} \qquad (47)$$

This differentiation was carried out on the assumption that the force contains Γ_i indirectly; that is, $\vec{F}_B$ contains $\vec{R}_B$, which in turn contains Γ_i. If $\vec{F}_B$ should contain Γ_i explicitly, a term $\left(N^T \frac{\partial \vec{F}_B}{\partial \Gamma_i}\right)_{explicit}$ must be added.

This is not clear in reference 1, and care must be exercised in its use.

The system of variational equations is, then, composed of equations (44) and (47) with, possibly, the term $\left(N^T \frac{\partial \vec{F}_B}{\partial \Gamma_i}\right)_{explicit}$ added to equation (47).

The initial conditions for this system of equations identify the values of $\frac{\partial \vec{R}_{IN}}{\partial \Gamma_i}$ and $\frac{\partial \dot{\vec{R}}_{IN}}{\partial \Gamma_i}$ at the initial time t_o.

Solving this system of equations will give $\frac{\partial \vec{R}_{IN}}{\partial \Gamma_i}$ as a function of time.

4.0 RESULTS

Pines's representation of the potential satisfies the requirement that the singularities in polar orbits be removed.

Attention is called, among the main formulas collected at the end of section 3.5.4, to equation (21), which was shown to be the most stable of the recurrence relations for the "derived" Legendre functions.

A preliminary computer subroutine was written, making full use of the recursion relations derived by Pines. It is contained in appendix D of this note.

James Kirkpatrick, of NASA, included this subroutine in place of the usual formulation of the potential in his program (which integrates the equations of motion with an Adams method). He noted a substantial improvement in the time of execution.

The subroutine, as shown in appendix D, makes liberal use of storage locations. After all the terms up to the desired order and degree have been generated and stored, several algebraic operations are performed upon them to obtain the components of the force. The terms remain in the same storage locations even after they have been used.

The subroutine has since been improved with a view to saving storage locations. The algebraic operations are performed as soon as the necessary terms are generated, and only those terms that will be of some future use are retained. This has been the work mostly of J. Kirkpatrick.

Although a complete evaluation of the gain in time and storage has not been made as yet, it is possible to say at this point that a model of the potential including terms up to $n = m = 18$ has been used with Pines's formulation and produced more integration steps in less time than a model with terms up to $n = m = 8$ with the usual formulation. The integrator in which these models were included was, of course, the same.

5.0 CONCLUSIONS

The formulation proposed by Pines is such that no singularities are present for any positions of the point.

The recursion relations are easy to program, and the program can be made in a way that will result in savings in storage and time of computation; the relations are stable and numerically efficient at all orders.

Generation of equations to study the effects on the attractive force of changes in the coefficients of the spherical harmonics or the effects on the position vector of changes in parameters such as those coefficients (and therefore the order of the potential model), initial position, or initial velocity is a simple procedure that makes use of the same recurrence relations.

APPENDIX A - GRADIENTS OF r, $\sin \alpha$, AND λ

APPENDIX A

GRADIENTS OF r, sin α, AND λ

Consider the transformation between Cartesian and spherical coordinates.

$$\begin{cases} x = r\cos\alpha\cos\lambda \\ y = r\cos\alpha\sin\lambda \\ z = r\sin\alpha \end{cases} \quad \text{and} \quad \begin{cases} r = (x^2+y^2+z^2)^{1/2} \\ \sin\alpha = \dfrac{z}{r} = \dfrac{z}{(x^2+y^2+z^2)^{1/2}} \\ \lambda = \arctan\dfrac{y}{x} \end{cases}$$

The gradients of r, $\sin\alpha$, and λ are, then,

$$\nabla r = \frac{\partial r}{\partial x}\hat{i} + \frac{\partial r}{\partial y}\hat{j} + \frac{\partial r}{\partial z}\hat{k}$$

$$= \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

$$= \frac{1}{r}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{1}{r}\vec{R} = \hat{R}$$

$$\nabla(\sin\alpha) = \frac{\partial(\sin\alpha)}{\partial x}\hat{i} + \frac{\partial(\sin\alpha)}{\partial y}\hat{j} + \frac{\partial(\sin\alpha)}{\partial z}\hat{k}$$

$$= -\frac{xz}{r^3}\hat{i} - \frac{yz}{r^3}\hat{j} + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)\hat{k}$$

$$= \frac{1}{r}\hat{k} - \frac{z}{r^3}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{1}{r}\hat{k} - \frac{1}{r}\frac{z}{r}\frac{\vec{R}}{r}$$

$$= \frac{1}{r}\hat{k} - \frac{\sin\alpha}{r}\hat{R}$$

$$\nabla\lambda = \frac{\partial\lambda}{\partial x}\hat{i} + \frac{\partial\lambda}{\partial y}\hat{j} + \frac{\partial\lambda}{\partial z}\hat{k}$$

$$= \frac{1}{1 + \frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)\hat{i} + \frac{1}{1 + \frac{y^2}{x^2}}\frac{1}{x}\hat{j} + 0\hat{k}$$

$$= -\frac{y}{x^2 + y^2}\hat{i} + \frac{x}{x^2 + y^2}\hat{j}$$

$$= -\frac{y}{r^2 - z^2}\hat{i} + \frac{x}{r^2 - z^2}\hat{j}$$

$$= -\frac{y}{r^2 - r^2\sin^2\alpha}\hat{i} + \frac{x}{r^2 - r^2\sin^2\alpha}\hat{j}$$

$$= -\frac{y}{r^2(1 - \sin^2\alpha)}\hat{i} + \frac{x}{r^2(1 - \sin^2\alpha)}\hat{j}$$

$$= \frac{1}{r^2(1 - \sin^2\alpha)}(-y\hat{j} + x\hat{i})$$

Now

$$\hat{k} \times \hat{R} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \end{bmatrix} = -\frac{y}{r}\hat{i} + \frac{x}{r}\hat{j}$$

So

$$\nabla\lambda = \frac{1}{r(1 - \sin^2\alpha)}\hat{k} \times \hat{R}$$

APPENDIX B - THE CAUCHY-RIEMANN CONDITIONS

APPENDIX B

THE CAUCHY-RIEMANN CONDITIONS

This derivation may be found, for instance, in reference 3. Let $\zeta = s + it$ be a complex variable and let $w = u(s,t) + iv(s,t)$ be a complex function of $\zeta, w = w(\zeta)$.

The function $w(\zeta)$ is said to be analytic at a point ζ if the derivative $\frac{dw}{d\zeta}$ exists and is finite at ζ and at all points of a neighborhood of ζ. The derivative is defined as

$$\frac{dw}{d\zeta} = \lim_{\Delta\zeta \to 0} \frac{w(\zeta + \Delta\zeta) - w(\zeta)}{\Delta\zeta}$$

where $\Delta\zeta = \Delta s + i\Delta t$.

If the limit is to exist, it must be independent of the way in which $\Delta\zeta$

$$\frac{dw}{d\zeta} = \lim_{\substack{\Delta s \to 0 \\ \Delta t \to 0}} \frac{[u(s+\Delta s,\ t+\Delta t) + iv(s+\Delta s,\ t+\Delta t)] - [u(s,t) + iv(s,t)]}{\Delta s + i\Delta t}$$

$$= \lim_{\substack{\Delta s \to 0 \\ \Delta t \to 0}} \left\{ \frac{\{[u(s+\Delta s,\ t+\Delta t) - u(s,t)] + i[v(s+\Delta s,\ t+\Delta t) - v(s,t)]\}\Delta s}{\Delta s^2 + \Delta t^2} - \frac{i\{[u(s+\Delta s,\ t+\Delta t) - u(s,t)] + i[v(s+\Delta s,\ t+\Delta t) - v(s,t)]\}\Delta t}{\Delta s^2 + \Delta t^2} \right\}$$

Now let $\Delta t \equiv 0$. Then

$$\frac{dw}{d\zeta} = \lim_{\Delta s \to 0} \left[\frac{u(s+\Delta s,t) - u(s,t)}{\Delta s} + i\, \frac{v(s+\Delta s,t) - v(s,t)}{\Delta s} \right] = \frac{\partial u}{\partial s} + i\frac{\partial v}{\partial s}$$

Let, now, $\Delta s \equiv 0$. Then

$$\frac{dw}{d\zeta} = \lim_{\Delta t \to 0} \left[-i\,\frac{u(s,t+\Delta t) - u(s,t)}{\Delta t} + \frac{v(s,t+\Delta t) - v(s,t)}{\Delta t} \right] = \frac{\partial v}{\partial t} - i\frac{\partial u}{\partial t}$$

Since equality of complex numbers requires equality of their real parts as well as of their imaginary parts, the result is

$$\boxed{\begin{aligned} \frac{\partial u}{\partial s} &= \frac{\partial v}{\partial t} \\ \frac{\partial u}{\partial t} &= -\frac{\partial v}{\partial s} \end{aligned}} \longrightarrow \text{ the Cauchy-Riemann conditions.}$$

APPENDIX C - ERRORS IN THE PAPER

APPENDIX C

ERRORS IN THE PAPER

Although reference 1 is, in general, correct and accurate, a few minor errors and misprints may lead the reader into difficulties.

These errors and misprints are to be found in equations (10), the last two of equations (25), equation (30a), and equations (39). The second of equations (45) is not very clear. These numbers refer to the equations in reference 1.

Equations (10) of reference 1 read

$$\cos m\lambda \cos m\alpha = r_m(s,t)$$

$$\sin m\lambda \cos m\alpha = i_m(s,t)$$

when they should be

$$\cos m\lambda \cos^m\alpha = r_m(s,t)$$

$$\sin m\lambda \cos^m\alpha = i_m(s,t)$$

There are four equations (25), of which only the last two are wrong. Since they are not used afterwards in the paper, the error has no consequences.

The functions r_m and i_m are homogeneous of degree m in s and t. Then Euler's theorem on homogeneous functions results in

$$s\frac{\partial r_m}{\partial s} + t\frac{\partial r_m}{\partial t} = mr_m$$

$$s\frac{\partial i_m}{\partial s} + t\frac{\partial i_m}{\partial t} = mi_m$$

These could be written, with the help of the Cauchy-Riemann conditions, as

$$s\frac{\partial r_m}{\partial s} - t\frac{\partial i_m}{\partial s} = mr_m$$

$$s\frac{\partial i_m}{\partial s} + t\frac{\partial r_m}{\partial s} = mi_m$$

These may be the equations that were to be presented. Instead, the last two of equations (25) of the reference read

$$s\frac{\partial r_m}{\partial s} - t\frac{\partial i_m}{\partial t} = r_m$$

$$s\frac{\partial i_m}{\partial s} + t\frac{dr_m}{\partial t} = i_m$$

It is easy to verify that these are wrong; an example will suffice. Let $m = 2$, for which $r_2 = s^2 - t^2$, $i_2 = 2ts$, $\frac{\partial r_2}{\partial s} = 2s$, $\frac{\partial i_2}{\partial t} = 2s$, $\frac{\partial r_2}{\partial t} = -2t$, and $\frac{\partial i_2}{\partial s} = 2t$, and obtain the results $2s^2 - 2st = s^2 - t^2$ and $-2st + 2t^2 = 2st$, which are obviously false.

Equation (30a) of reference 1 just lacks a minus sign; this is corrected in equation (30b).

Equations (39) of reference 1 have several mistakes. As shown in section 3.7 of this note, the derivatives are

$$f_{C_{N,M,1}} = \frac{\partial a_1}{\partial C_{N,M}} + s\frac{\partial a_4}{\partial C_{N,M}} \quad ; \quad f_{S_{N,M,1}} = \frac{\partial a_1}{\partial S_{N,M}} + s\frac{\partial a_4}{\partial S_{N,M}}$$

$$f_{C_{N,M,2}} = \frac{\partial a_2}{\partial C_{N,M}} + t\frac{\partial a_4}{\partial C_{N,M}} \quad ; \quad f_{S_{N,M,2}} = \frac{\partial a_2}{\partial S_{N,M}} + t\frac{\partial a_4}{\partial S_{N,M}}$$

$$f_{C_{N,M,3}} = \frac{\partial a_3}{\partial C_{N,M}} + u\frac{\partial a_4}{\partial C_{N,M}} \quad ; \quad f_{S_{N,M,3}} = \frac{\partial a_3}{\partial S_{N,M}} + u\frac{\partial a_4}{\partial S_{N,M}}$$

These derivatives are to be obtained from the expressions of a_1, a_2, a_3, and a_4; namely,

$$a_1 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}(C_{n,m}r_{m-1} + S_{n,m}i_{m-1})$$

$$a_2 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m}(S_{n,m}r_{m-1} - C_{n,m}i_{m-1})$$

$$a_3 = \sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n,m+1}(C_{n,m}r_m + S_{n,m}i_m)$$

$$a_4 = -\sum_{n=0}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} A_{n+1,m+1}(C_{n,m}r_m + S_{n,m}i_m)$$

APPENDIX C

Consider a specific pair of coefficients $C_{N,M}$ and $S_{N,M}$. They appear only once in the expressions of a_1, a_2, a_3, and a_4. The term a_1 may be written so as to show separately the part containing these coefficients.

$$a_1 = \sum_{n=0}^{N-1} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m} E_{n,m} + \frac{\rho_{N+1}}{a} \sum_{m=0}^{M-1} mA_{N,m} E_{N,m}$$

$$+ \frac{\rho_{N+1}}{a} MA_{N,M} C_{N,M} r_{M-1} + \frac{\rho_{N+1}}{a} MA_{N,M} S_{N,M} i_{M-1}$$

$$+ \frac{\rho_{N+1}}{a} \sum_{m=M+1}^{n} mA_{N,m} E_{N,m} + \sum_{n=N+1}^{\infty} \frac{\rho_{n+1}}{a} \sum_{m=0}^{n} mA_{n,m} E_{n,m}$$

Similarly for a_2, a_3, and a_4 (but showing only the pertinent parts),

$$a_2 = \cdots + \frac{\rho_{N+1}}{a} MA_{N,M} S_{N,M} r_{M-1} - \frac{\rho_{N+1}}{a} MA_{N,M} C_{N,M} i_{M-1} + \cdots$$

$$a_3 = \cdots + \frac{\rho_{N+1}}{a} A_{N,M+1} C_{N,M} r_M + \frac{\rho_{N+1}}{a} A_{N,M+1} S_{N,M} i_M + \cdots$$

$$a_4 = \cdots - \frac{\rho_{N+1}}{a} A_{N+1,M+1} C_{N,M} r_M - \frac{\rho_{N+1}}{a} A_{N+1,M+1} S_{N,M} i_M + \cdots$$

with the result that

$$\frac{\partial a_1}{\partial C_{N,M}} = \frac{\rho_{N+1}}{a} MA_{N,M} r_{M-1} \quad ; \quad \frac{\partial a_1}{\partial S_{N,M}} = \frac{\rho_{N+1}}{a} MA_{N,M} i_{M-1}$$

$$\frac{\partial a_2}{\partial C_{N,M}} = -\frac{\rho_{N+1}}{a} MA_{N,M} i_{M-1} \quad ; \quad \frac{\partial a_2}{\partial S_{N,M}} = \frac{\rho_{N+1}}{a} MA_{N,M} r_{M-1}$$

$$\frac{\partial a_3}{\partial C_{N,M}} = \frac{\rho_{N+1}}{a} A_{N,M+1} r_M \quad ; \quad \frac{\partial a_3}{\partial S_{N,M}} = \frac{\rho_{N+1}}{a} A_{N,M+1} i_M$$

$$\frac{\partial a_4}{\partial C_{N,M}} = -\frac{\rho_{N+1}}{a} A_{N+1,M+1} r_M ; \quad \frac{\partial a_4}{\partial S_{N,M}} = -\frac{\rho_{N+1}}{a} A_{N+1,M+1} i_M$$

Knowing these, it is an easy matter to arrive at the correct equations, which may be found in section 3.7 of this note as equations (40).

The lack of clarity of the second of equations (45) of reference 1 has been discussed at the end of section 3.7 of the present note.

APPENDIX D - A COMPUTER PROGRAM IMPLEMENTATION

APPENDIX D

A COMPUTER PROGRAM IMPLEMENTATION

```
      SUBROUTINE GRAVPOT (NAXO,X,Y,Z,FX,FY,FZ)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION CREAL(NAXO), CIMAG(NAXO), RHO(NAXO),
     1 A(NAXO,NAXO),D(NAXO,NAXO),E(NAXO,NAXO),F(NAXO,NAXO)
      COMMON/CONST/C(NAXO,NAXO),S(NAXO,NAXO),ZONAL(NAXO),
     1 TERRAD, TMU
C
C     NAXO IS MAXO+1, WHERE MAXO IS THE ORDER TO WHICH THE
C     POTENTIAL IS DESIRED.
C
C     TERRAD IS THE EARTH'S RADIUS, MU THE GRAVITATIONAL CONSTANT,
C     C(N,M), S(N,M) AND ZONAL(N) THE COEFFICIENTS OF THE
C     SPHERICAL HARMONICS, A(N,M) THE DERIVED LEGENDRE
C     FUNCTIONS, X, Y AND Z THE COORDINATES OF THE POINT,
C     D(N,M), E(N,M) AND F(N,M) THE FUNCTIONS OF THE
C     COMPLEX VARIABLE ZETA=CREAL+I*CIMAG AND
C     RHO IS THE FUNCTION OF THE RADIAL DISTANCE.  FX,
C     FY AND FZ ARE THE COMPONENTS OF THE FORCE.
C
C     GET THE DIRECTION COSINES ESS, T AND U.
C
      MAXO=NAXO-1
      RINV=1.0D0/SQRT(X*X+Y*Y+Z*Z)
      ESS=X*RINV
      T=Y*RINV
      U=Z*RINV
C
C     GENERATE THE FUNCTIONS CREAL, CIMAG, A, D, E, F AND RHO.
C
      RO=TERRAD*RINV
      RHOZERO=TMU*RHOZERO
      RHO(1)=RO*RHOZERO
      CREAL(1)=ESS
      CIMAG(1)=T
      D(1,1)=0.0D0
      E(1,1)=0.0D0
      F(1,1)=0.0D0
      A(1,1)=1.0D0
      DO 1  I=2,NAXO
      RHO(I)=RO*RHO(I-1)
```

```
      CREAL(I)=ESS*CREAL(I-1)-T*CIMAG(I-1)
      CIMAG(I)=ESS*CIMAG(I-1)+T*CREAL(I-1)
      D(I,1)=ESS*C(I,1)+T*S(I,1)
      E(I,1)=C(I,1)
      F(I,1)=S(I,1)
      A(I,I)=(2*I-1)*A(I-1,I-1)
      A(I,I-1)=U*A(I,I)
      DO 1  K=2,I
      IF (I.EQ.NAXO) GO TO 9
      D(I,K)=C(I,K)*CREAL(K)+S(I,K)*CIMAG(K)
      E(I,K)=C(I,K)*CREAL(K-1)+S(I,K)*CIMAG(K-1)
      F(I,K)=S(I,K)*CREAL(K-1)-C(I,K)*CIMAG(K-1)
    9 CONTINUE
      IF (I.EQ.2) GO TO 10
      L=I-2
      DO 1  J=1,L
      A(I,I-J-1)=(U*A(I,I-J)-A(I-1,I-J))/(J+1)
    1 CONTINUE
   10 CONTINUE
C     NOW COMPUTE THE AUXILIARY QUANTITIES A1, A2, A3 AND
C     A4, NEEDED TO FIND THE COMPONENTS OF THE FORCE.
      A1=0.0D0
      A2=0.0D0
      A3=0.0D0
      A4=RHOZERO*RINV
      DO 2  N=2,MAXO
      FAC1=0.0D0
      FAC2=0.0D0
      FAC3=A(N,1)*ZONAL(N)
      FAC4=A(N+1,1)*ZONAL(N)
      DO 3  M=1,N
      FAC1=FAC1+M*A(N,M)*E(N,M)
      FAC2=FAC2+M*A(N,M)*F(N,M)
      FAC3=FAC3+A(N,M+1)*D(N,M)
    3 FAC4=FAC4+A(N+1,M+1)*D(N,M)
      A1=A1+RINV*RHO(N)*FAC1
      A2=A2+RINV*RHO(N)*FAC2
      A3=A3+RINV*RHO(N)*FAC3
      A4=A4+RINV*RHO(N)*FAC4
    2 CONTINUE
      FX=A1-ESS*A4
      FY=A2-T*A4
      FZ=A3-U*A4
      RETURN
      END
```

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